$U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory and Z_4 parafermion fractional quantum Hall states

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We study $U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory with integral coupling constants (k,l) and its relation to certain non-Abelian fractional quantum Hall (FQH) states. For the $U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory, we show how to compute the dimension of its Hilbert space on genus g surfaces and how this yields the quantum dimensions of topologically distinct excitations. We find that Z_2 vortices in the $U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory carry non-Abelian statistics and we show how to compute the dimension of the Hilbert space in the presence of n pairs of Z_2 vortices on a sphere. These results allow us to show that $l=3$ $U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory is the low-energy effective theory for the Z_4 parafermion (Read-Rezayi) fractional quantum Hall states, which occur at filling fraction $\nu = \frac{2}{2k-3}$. The $U(1) \times U(1) \rtimes Z_2$ theory is more useful than an alternative $SU(2)_4 \times U(1)/U(1)$ Chern-Simons theory because the fields are more closely related to physical degrees of freedom of the electron fluid and to an Abelian bilayer phase on the other side of a two-component to single-component quantum phase transition. We discuss the possibility of using this theory to understand further phase transitions in FQH systems, especially the $\nu = 2/3$ phase diagram.

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I. INTRODUCTION

One of the most exciting breakthroughs in condensed-matter physics has been the discovery that there exist quantum phases of matter at zero temperature that cannot be described by their pattern of symmetry breaking.[1] The prototypical and perhaps most well-studied examples of these phases are the fractional quantum Hall (FQH) states,[2] which exhibit a different kind of order, called topological order.[3] Topologically ordered phases are currently the subject of intense interest because of the possibility of detecting excitations that exhibit non-Abelian statistics,[4,5] and subsequently manipulating these non-Abelian excitations for robust quantum information storage and processing.[6–8]

One way to improve our understanding of topological order in the fractional quantum Hall states is to study phase transitions between states with different topological orders. While much is known about phase transitions between phases with different patterns of symmetry breaking, much less is known about phase transitions between phases with different topological orders. Aside from its intrinsic interest, such information may be useful in identifying the topological order of a certain FQH state, which is currently a significant challenge. The experimental observation of a continuous phase transition in a FQH system may help us identify the topological order of one of the phases if we know theoretically which topologically ordered phases can be connected to each other through a continuous phase transition and which cannot. Ultimately, we would like to have an understanding of all of the possible topological orders in FQH states and how they can be related to each other through continuous phase transitions.

We may hope to understand a phase transition between two phases if we have a field theory that describes each phase and we know how the field theories of the two phases are related to each other. In the case of the fractional quantum Hall states, it is well known that the long-distance low-energy behavior is described by certain topological field theories in 2+1 dimensions,[9] called Chern-Simons theories. For the Laughlin states and other Abelian FQH states, such as the Halperin states, the hierarchy states, and Jain states, the long-wavelength behavior is described by Chern-Simons theories with a number of $U(1)$ gauge fields.[9–11]

For the non-Abelian FQH states, the corresponding Chern-Simons (CS) theory has a non-Abelian gauge group.[12,13] The most well-studied examples of non-Abelian FQH states are the Moore-Read Pfaffian state[4] and some of its generalizations, the Read-Rezayi (or Z_k parafermion) states.[14] The bosonic $\nu=1$ Pfaffian is described by $SU(2)_2$ Chern-Simons theory,[13] or alternatively, by $SO(5)_1$ Chern-Simons theory,[12] while the effective theories for the other states are less well understood. It has been proposed that the Read-Rezayi Z_k parafermion states are described by $SU(2)_k \times U(1)/U(1)$ Chern-Simons theory.[15]

In this paper, we show that Chern-Simons theory with gauge group $U(1) \times U(1) \rtimes Z_2$ describes the long-wavelength properties of the Z_4 parafermion Read-Rezayi FQH state. The significance of this result is that there is a bilayer state, the $(k,k,k-3)$ Halperin state at $\nu = \frac{2}{2k-3}$, which may undergo a bilayer to single-layer quantum phase transition to the Z_4 parafermion state as the interlayer tunneling is increased.[16] The bilayer phase is described by a $U(1) \times U(1)$ Chern-Simons theory. This new formulation of the Chern-Simons theory for the Z_4 parafermion state may therefore be useful in understanding the phase transition because the gauge groups $U(1) \times U(1) \rtimes Z_2$ and $U(1) \times U(1)$ are closely related, and because the fields in the $U(1) \times U(1) \rtimes Z_2$ theory are more closely related to physical degrees of freedom of the electron fluid than they are in the proposed alternative $SU(2)_4 \times U(1)/U(1)$ theory.

In addition to aiding us in understanding this phase transition, this study shows how to compute concretely various topological properties of a Chern-Simons theory with a disconnected gauge group. For Chern-Simons theories at level k, where the gauge group is a simple Lie group G, there is a straightforward prescription to compute topological proper-

ties. The different quasiparticles are labeled by the integrable highest weight representations of the affine Lie algebra $\hat{g}_k$, where g is the Lie algebra of G, while the quasiparticle fusion rules are given by the Clebsch-Gordon coefficients of the integrable representations of $\hat{g}_k$.[17] In contrast, when the gauge group is disconnected and is of the form $G \rtimes H$, where H is a discrete automorphism group of G, it is much less straightforward to compute the topological properties of the Chern-Simons theory directly. One reason for this is that discrete gauge theories are most easily studied (and defined) on a lattice, while it is difficult to formulate lattice versions of Chern-Simons theories. This complicates the study of Chern-Simons theories with disconnected gauge groups.

In the case where the gauge group is $U(1) \times U(1) \rtimes Z_2$, we show how to compute the ground-state degeneracy on genus g surfaces and how this yields the quantum dimensions of the quasiparticles. We find that the Z_2 vortices carry non-Abelian statistics and we show how to compute the degeneracy of states in the presence of n pairs of Z_2 vortices. The results, for a certain choice of coupling constants, agree exactly with results obtained in other ways for the Z_4 parafermion FQH state.

II. MOTIVATION AND BACKGROUND

One interesting way of obtaining the Pfaffian quantum Hall states is by starting with a bilayer $(k,k,k-2)$ quantum Hall state and taking the interlayer tunneling to infinity. The bilayer state is at a filling fraction $\nu = \frac{1}{k-1}$ and is described by the wave function $\Psi = \Phi(\{z_i\}, \{w_i\}) \exp[-\frac{1}{4}\Sigma_i(|z_i|^2 + |w_i|^2)]$, with

$$\Phi = \prod_{i<j}^{N}(z_i - z_j)^k \prod_{i<j}^{N}(w_i - w_j)^k \prod_{i,j}^{N}(z_i - w_j)^{k-2}. \quad (1)$$

Here, $z_i = x_i + iy_i$ is the complex coordinate of the ith electron in one layer and w_i is the complex coordinate for the ith electron in the other layer.

As the tunneling is taken to infinity, we effectively end up with a single-layer state. The particles in the two layers become indistinguishable and so we might expect that the resulting wave function is the $(k,k,k-2)$ bilayer wave function but (anti)symmetrized between the $\{z_i\}$ and $\{w_i\}$ coordinates. The resulting wave function happens to be the Pfaffian state,

$$\Psi_{Pf}(\{z_i\}) = Pf\left(\frac{1}{z_i - z_j}\right) \prod_{i<j}^{2N}(z_i - z_j)^{k-1} = S\{\Psi(\{z_i\}, \{w_i\})\}, \quad (2)$$

where $S\{\cdots\}$ refers to symmetrization or antisymmetrization over z_i and w_i depending on whether the particles are bosons are fermions. Here we have set $z_{N+i} = w_i$. Indeed, the $(k,k,k-2)$ bilayer states undergo a continuous quantum phase transition to the single-layer $\nu = \frac{1}{k-1}$ Pfaffian states as the interlayer tunneling is increased.[18,19]

In a similar fashion, the $(k,k,k-3)$ bilayer wave functions, when (anti)symmetrized over the coordinates of particles in the two layers, yield the Z_4 parafermion states at filling fraction $\nu = \frac{2}{2k-3}$.[14,16] One way to verify this statement is through an operator algebra approach that also naturally

suggests $U(1) \times U(1) \rtimes Z_2$ as the appropriate gauge group for the corresponding Chern-Simons theory (see Appendix A). This observation suggests that as the interlayer tunneling is increased, there may be a region of the phase diagram where there is a phase transition from the bilayer $(k,k,k-3)$ state to the single-layer non-Abelian Z_4 parafermion state. For $k=3$, this is a phase transition at $\nu = 2/3$, the phase diagram of which has attracted both theoretical and experimental attention.

Given this perspective, we might expect that we can understand the low-energy effective field theory of the Pfaffian and Z_4 parafermion states by gauging a discrete Z_2 symmetry associated with the Z_2 symmetry of interchanging the two layers. The effective field theories for the bilayer states are the $U(1) \times U(1)$ Chern-Simons theories with the field strength of one $U(1)$ gauge field describing the electron density for one layer and the field strength of the other gauge field for the other layer. This perspective suggests that the topological properties of these non-Abelian states can be described by a $U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory. This is a $U(1) \times U(1)$ Chern-Simons theory with an additional local Z_2 gauge symmetry. The semidirect product $\rtimes$ here indicates that the Z_2 acts on the group $U(1) \times U(1)$; the Z_2 group element does not commute with elements of $U(1) \times U(1)$. In other words, elements of the group are (a,ρ), where $a \in U(1) \times U(1)$ and $\rho \in Z_2$, and multiplication is defined by $(a_1,\rho_1)*(a_2,\rho_2) = (a_1\rho_1 a_2\rho_1, \rho_1\rho_2)$. This expectation for $U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory turns out to be correct for the Z_4 parafermion states but not quite correct for the Pfaffian states, as we will discuss.

We already have a field theory that correctly describes the topological properties of the bosonic $\nu = 1$ Pfaffian quantum Hall state. This is the $SU(2)_2$ Chern-Simons theory described in Ref. 13 or the $SO(5)_1$ Chern-Simons theory described in Ref. 12. [The Pfaffian quantum Hall state at other filling fractions are described by $SU(2)_2 \times U(1)/U(1)$ or $SO(5)_1 \times U(1)/U(1)$ Chern-Simons theory.[12]] Similarly, the $SU(2)_k \times U(1)/U(1)$ Chern-Simons theories described in Ref. 15 encapsulate in some sense the topological properties of the Z_k parafermion states. A possible shortcoming of those theories, however, is that it can be unclear how to connect the degrees of freedom of the field theory to the physical degrees of freedom of the electron liquid. In contrast, the $U(1) \times U(1) \rtimes Z_2$ makes clearer the connection between the gauge fields and various physical degrees of freedom. It also makes clearer the relation to the bilayer state on the other side of the phase transition. Given this closer contact to the physical degrees of freedom of the electron fluid and to the bilayer Abelian phase, it is possible that this point of view may aid us in understanding physical properties of these quantum Hall states, such as the quantum phase transition between two topologically ordered phases: the bilayer Abelian phases and the non-Abelian single-layer phases.

The fact that such a Chern-Simons theory might describe the Pfaffian and/or Z_4 parafermion FQH states might also be expected from another point of view. It is known that the Z_4 parafermion conformal field theory (CFT), which is used in constructing the Z_4 parafermion FQH states, is dual to the rational Z_2 orbifold at a certain radius.[20] The rational Z_2 orbifold at radius R is the theory of a scalar boson φ compac-

tified on a circle of radius R, i.e., $\varphi \sim \varphi + 2\pi R$, and that is gauged by a Z_2 action $\varphi \sim -\varphi$. Furthermore, the Z_2 orbifold at a different radius is dual to two copies of the Ising CFT, which is used to construct the Pfaffian states. The Chern-Simons theory corresponding to the Z_2 orbifold CFT has gauge group $O(2)$, which we can think of as $U(1) \rtimes Z_2$.[21] This line of thinking is what led the authors of Ref. 22 to first mention that $U(1) \times O(2)$ Chern-Simons theories are related to the Pfaffian and Z_4 parafermion states. In the Z_4 parafermion case, the relation to $U(1) \times O(2)$ is suggestive but incomplete because the $U(1)$ and the $O(2)$ need to be "glued" together in an appropriate way; we elaborate more on this point in Appendix B. The proper formulation is the $U(1) \times U(1) \rtimes Z_2$ theory that we present here and for which we compute many topological properties.

Let us first discuss the $U(1) \times U(1)$ Chern-Simons theories that describe the $(k,k,k-l)$ bilayer states. These are defined by the Lagrangian

$$L = \frac{k}{4\pi} \int_M (a\partial a + \tilde{a}\partial\tilde{a}) + \frac{k-l}{4\pi} \int_M (a\partial\tilde{a} + \tilde{a}\partial a), \quad (3)$$

where M is a two-dimensional manifold and $a(x,y,t)$ and $\tilde{a}(x,y,t)$ are the two $U(1)$ gauge fields defined on $M \times \mathbb{R}$. M describes space and $\mathbb{R}$ describes time. The electron current density in the top and bottom layers, j_μ and $\tilde{j}_\mu$, respectively, are given by

$$j_\mu = \frac{1}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu a_\lambda,$$

$$\tilde{j}_\mu = \frac{1}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu \tilde{a}_\lambda. \quad (4)$$

In the $U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory, we package the two gauge fields in the following way:

$$A_\mu = \begin{pmatrix} a_\mu & 0 \\ 0 & \tilde{a}_\mu \end{pmatrix}. \quad (5)$$

The gauge group $G = U(1) \times U(1) \rtimes Z_2$ consists of the $U(1) \times U(1)$ part, which we can write as

$$U = \begin{pmatrix} e^{if} & 0 \\ 0 & e^{ig} \end{pmatrix}, \quad (6)$$

and the Z_2 part, which contains the identity and the nontrivial element σ_1,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (7)$$

Thus, in addition to the usual $U(1) \times U(1)$ gauge symmetry associated with the two gauge fields, there is a local Z_2 gauge symmetry, which can be thought of in the following way. The space of physical configurations at a certain space-time point (x,y,t) is to be described by the unordered pair $[a_\mu(x,y,t), \tilde{a}_\mu(x,y,t)]$. The action of the Z_2 is to interchange $a_\mu(x,y,t)$ and $\tilde{a}_\mu(x,y,t)$ at the point (x,y,t). Physically, we may perhaps envision this as an electron from one layer and an electron from the other layer being interchanged. In order

to define a sensible action, we need to be dealing with differentiable gauge fields. So, we require the gauge fields to be smooth functions on M, thus automatically gauge fixing the local Z_2 and leaving behind a residual global Z_2 symmetry associated with interchanging a and $\tilde{a}$ at every point in space-time. In this sense, we can use the action given by Eq. (3) to describe our $U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory.

Although the $U(1) \times U(1)$ Chern-Simons theory and $U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory formally share the same Lagrangian, their gauge structure is different. This is why the same Lagrangian actually describes two different theories. This example demonstrates that the Lagrangian is not a good symbol for a one-to-one labeling of different topological field theories.

III. GROUND-STATE DEGENERACY FOR $U(1) \times U(1) \rtimes Z_2$ CHERN-SIMONS THEORY

The first check that a field theory correctly describes a given topologically ordered phase is whether it correctly reproduces the ground-state degeneracy of the system on surfaces of higher genus. Accordingly, we begin our study of $U(1) \times U(1) \rtimes Z_2$ by calculating the ground-state degeneracy on a torus. We then calculate the degeneracy on surfaces of arbitrary genus, from which we deduce the quantum dimensions of the quasiparticles. Finally, we study the quasiparticles.

Gauge theory with gauge group G on a manifold M is most generally defined by starting with a principal G bundle on M and defining the gauge field, a Lie algebra-valued one-form, as a connection on the bundle. Often, one is concerned with situations in which $M = \mathbb{R}^n$, in which case there is a global coordinate system and the gauge field can be written in coordinates everywhere as $a_\mu dx^\mu$, where a_μ is a Lie algebra-valued function on $\mathbb{R}^n$. In these cases, we do not need to be concerned with the more general fiber bundle definition in order to compute quantities of interest. The situation is more complicated in general, when M does not have a global coordinate system, in which case we can only locally define $a = a_\mu dx^\mu$ in any given coordinate chart. In these situations, it is often convenient, when possible, to view the gauge field as a function defined on $\mathbb{R}^n$, where n is the dimension of M, and to impose suitable periodicity conditions. This allows us to work in a global coordinate system and may simplify certain computations. For example, for $U(1)$ gauge theory on a torus, we can choose to work with a gauge field $a_\mu(x,y)$ defined over $\mathbb{R}^2$, but with periodic boundary conditions,

$$a_\mu(x,y) = a_\mu(x+L_x,y) = a_\mu(x,y+L_y). \quad (8)$$

In the case where $G = U(1) \times U(1) \rtimes Z_2$, the Z_2 gauge symmetry allows for the possibility of twisted sectors: configurations in which the gauge field is periodic up to conjugacy by an element of Z_2. On a torus, there are four sectors and the ground-state degeneracy is controlled by the degeneracy within each sector. In more mathematical terms, there are four distinct classes of $U(1) \times U(1) \rtimes Z_2$ bundles on a torus, distinguished by the four possible elements in the group $[\text{Hom}: \pi_1(T^2) \to Z_2]/Z_2$, which is the group of homomor-

phisms from the fundamental group of T^2 to Z_2, mod Z_2. Thus, we can think of $A_\mu(x,y,t)$ as defined on $\mathbb{R}^3$, with the following periodicity conditions:

$$A_\mu(x+L_x,y) = \sigma_1^{\epsilon_x} A_\mu(x,y) \sigma_1^{\epsilon_x},$$

$$A_\mu(x,y+L_y) = \sigma_1^{\epsilon_y} A_\mu(x,y) \sigma_1^{\epsilon_y}, \tag{9}$$

where ϵ_x and ϵ_y can each be 0 (untwisted) or 1 (twisted). Furthermore, in each of these sectors, the allowed gauge transformations $U(x,y)$ take the form (time index is suppressed)

$$U(x,y) = \begin{pmatrix} e^{if(x,y)} & 0 \\ 0 & e^{ig(x,y)} \end{pmatrix} \tag{10}$$

and must preserve the boundary conditions on A_μ,

$$U(x+L_x,y) = \sigma_1^{\epsilon_x} U(x,y) \sigma_1^{\epsilon_x},$$

$$U(x,y+L_y) = \sigma_1^{\epsilon_y} U(x,y) \sigma_1^{\epsilon_y}. \tag{11}$$

These transform A_μ in the usual way,

$$A_\mu \rightarrow U A_\mu U^{-1} + iU\partial_\mu U^{-1}. \tag{12}$$

The formulation of the theory on higher genus surfaces is similar. On a genus g surface, there are 2^{2g} different sectors, characterized by whether there is a Z_2 twist along various noncontractible loops. Across these twists, the two gauge fields a and $\tilde{a}$ transform into each other. The gauge transformations also obey these same twisted boundary conditions; this implies that the boundary conditions on the gauge fields are preserved under gauge transformations. The connection between this formulation and the definition of a principal G bundle on a compact Riemann surface can be made more precise by considering local coordinate charts, transition functions, etc., but here we do not pursue any further mathematical precision.

A. Ground-state degeneracy on a torus

As mentioned above, there are four sectors on a torus, one untwisted sector and three twisted sectors. We now proceed to compute the ground-state degeneracy in each sector. We follow the approach in Ref. 23, which was applied to continuous and connected gauge groups.

1. Untwisted sector

In the untwisted sector, the ground states are the Z_2 invariant states of a $U(1) \times U(1)$ Chern-Simons theory with the Lagrangian of Eq. (3). We partially fix the gauge by setting $a_0 = \tilde{a}_0 = 0$. The equations of motion for a_0 and $\tilde{a}_0$ act as constraints that require zero field strength: $f = \partial_x a_y - \partial_y a_x = 0$ and $\tilde{f} = \partial_x \tilde{a}_y - \partial_y \tilde{a}_x = 0$. This implies that gauge-inequivalent configurations are completely specified by the holonomies of the gauge fields around noncontractible loops of the torus, $\oint a \cdot dl$ and $\oint \tilde{a} \cdot dl$. This is a special case of the more general statement that flat G bundles are characterized by $[\mathrm{Hom}: \pi_1(M) \rightarrow G]/G$. We can parametrize this configuration space in the following way:

$$a_1(x,y,t) = \frac{2\pi}{L} X(t), \quad \tilde{a}_1(x,y,t) = \frac{2\pi}{L} \tilde{X}(t),$$

$$a_2(x,y,t) = \frac{2\pi}{L} Y(t), \quad \tilde{a}_2(x,y,t) = \frac{2\pi}{L} \tilde{Y}(t). \tag{13}$$

The large gauge transformations $a \rightarrow a + iU^{-1}\partial U$ with $U(x,y) = e^{2\pi imx/L + 2\pi iny/L}$ take $(X,Y) \rightarrow (X+m, Y+n)$. Thus (X,Y) and $(\tilde{X}, \tilde{Y})$ take values on a torus. Substitution into the action yields, up to total time derivatives,

$$L = 2\pi k(X\dot{Y} + \tilde{X}\dot{\tilde{Y}}) + 2\pi(k-l)(\tilde{X}\dot{Y} + X\dot{\tilde{Y}}). \tag{14}$$

The Hamiltonian vanishes. The momenta conjugate to Y and $\tilde{Y}$ are

$$p_Y = \frac{\delta L}{\delta \dot{Y}} = 2\pi kX + 2\pi(k-l)\tilde{X},$$

$$p_{\tilde{Y}} = \frac{\delta L}{\delta \dot{\tilde{Y}}} = 2\pi k\tilde{X} + 2\pi(k-l)X. \tag{15}$$

The wave functions for this system can be written as a sum of plane waves,

$$\psi(Y,\tilde{Y}) = \sum_{n,m} c_{n,m} e^{i2\pi nY + i2\pi m\tilde{Y}}. \tag{16}$$

In momentum space, the wave function becomes

$$\phi(p_Y, p_{\tilde{Y}}) = \sum_{n,m} c_{n,m} \delta(p_Y - 2\pi n)\delta(p_{\tilde{Y}} - 2\pi m), \tag{17}$$

or, equivalently,

$$\varphi(X,\tilde{X}) = \sum_{n,m} c_{n,m} \delta[kX + (k-l)\tilde{X} - n]\delta[k\tilde{X} + (k-l)X - m]. \tag{18}$$

Using the fact that $X \sim X+1$ and $\tilde{X} \sim \tilde{X}+1$, we find that

$$c_{n,m} = c_{n-k,m-k+l} = c_{n-k+l,m-k}. \tag{19}$$

There are $|l(2k-l)|$ independent coefficients $c_{n,m}$, which explains why the $(k,k,k-l)$ quantum Hall state has a degeneracy of $|l(2k-l)|$ on a torus.

We can label the quantum states by $|n,m\rangle$. The ground states in our $U(1) \times U(1) \rtimes Z_2$ theory will be the Z_2 invariant subspace of this Hilbert space; it will contain the diagonal states $|n,n\rangle$ and ones of the form $|n,m\rangle + |m,n\rangle$. A simple count of the Z_2 invariant states, using identifications (19), yields a total of

$$(|l| + 1)|2k - l|/2 \tag{20}$$

states in this untwisted sector.

2. Twisted sectors

There are three Z_2 twisted sectors, corresponding to twisting in either the x direction, the y direction, or both. Since modular transformations, i.e., diffeomorphisms that are not

continuously connected to the identity, are symmetries that can take one twisted sector to another, we expect that all twisted sectors should have the same degeneracy. This can be verified explicitly by computing the degeneracy in each case. Here we will only consider the case where the gauge fields are twisted in the y direction. More precisely this means that the gauge fields obey the following boundary conditions:

$$a_i(x, y+L) = \tilde{a}_i(x,y), \quad \tilde{a}_i(x, y+L) = a_i(x,y),$$

$$a_i(x+L, y) = a_i(x,y), \quad \tilde{a}_i(x+L, y) = \tilde{a}_i(x,y). \quad (21)$$

Given these twisted boundary conditions, we can consider a new field $c_\mu(x,y)$ defined on a space that is doubled in length in the y direction,

$$c_\mu(x,y) = \begin{cases} a_\mu(x,y), & 0 \le y \le L \\ \tilde{a}_\mu(x, y-L), & L \le y \le 2L. \end{cases} \quad (22)$$

Observe that c has the periodicity

$$c_\mu(x,y) = c_\mu(x+L, y) = c_\mu(x, y+2L). \quad (23)$$

The allowed gauge transformations that act on c_i are of the form $W(x,y) = e^{ih(x,y)}$, where $W(x,y)$ need only be periodic on the doubled torus,

$$W(x+L, y) = W(x, y+2L) = W(x,y), \quad (24)$$

c transforms as a typical $U(1)$ gauge field,

$$c \to c - \partial h. \quad (25)$$

In particular, there are large gauge transformations $W(x,y) = e^{i(2\pi m/L)x + i(2\pi n/2L)y}$ that change the zero mode of c_i,

$$c_i \to c_i + \frac{2\pi m}{L} + \frac{2\pi n}{2L}. \quad (26)$$

In terms of c, the Lagrangian becomes

$$L = \int_0^L dx \int_0^{2L} dy \left[\frac{k}{4\pi} c \partial c + \frac{k-l}{4\pi} c(x,y) \partial c(x, y-L) \right]. \quad (27)$$

Note that this Lagrangian is actually nonlocal in the field c, but this does not pose any additional difficulty. We can set temporal gauge $c_0 = 0$, i.e., $a_0 = \tilde{a}_0 = 0$, and view the equation of motion for c_0 as a constraint that forces the field strength for c to be zero. Thus, the gauge-inequivalent configurations can be parametrized as

$$c_i(x,y,t) = \frac{2\pi}{L_i} X_i(t), \quad (28)$$

where $L_1 = L$ and $L_2 = 2L$. Inserting this expansion into the Lagrangian gives, up to total time derivatives,

$$L = 2\pi(2k-l) X_1 \dot{X}_2. \quad (29)$$

Due to the existence of the large gauge transformations, we find that the zero modes X_i take values on a torus,

$$(X_1, X_2) \sim (X_1 + 1, X_2) \sim (X_1, X_2 + 1). \quad (30)$$

Thus, using the same techniques used in the previous section, we conclude that the ground-state degeneracy in this sector is $2k - l$. There are three different twisted sectors, so we find in total

$$3|2k - l| \quad (31)$$

states in the twisted sectors of the $U(1) \times U(1) \rtimes Z_2$ theory.

3. Total ground-state degeneracy on torus

Adding the degeneracies from the twisted and the untwisted sectors, we find that the total ground-state degeneracy on a torus in the $U(1) \times U(1) \rtimes Z_2$ theory is

$$\text{ground-state Deg. on torus} = (|l| + 7)|2k - l|/2. \quad (32)$$

For $l = 2$ and $k > 1$, the filling fraction is $\nu = \frac{1}{k-1}$ and the above formula gives $9(k-1)$ states on a torus. Compare this to the torus degeneracy of the $\nu = \frac{1}{k-1}$ Pfaffian state, which is $3(k-1)$. We see that the $U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory for $l = 2$ has a torus ground-state degeneracy that is three times that of the Pfaffian state. So the $U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory for $l = 2$ cannot directly describe the Pfaffian state. In Appendix B, we argue that, for $l = 2$, $U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory describes the Pfaffian state plus an extra copy of the Ising model.

For $l = 3$ and $k > 2$, the filling fraction is $\nu = \frac{2}{2k-3}$ and Eq. (32) gives $5(2k-3)$ ground states on a torus. The $\nu = \frac{2}{2k-3}$ Z_4 parafermion state also gives rise to same torus degeneracy of $5(2k-3)$. Thus, we would like to propose that the $U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory for $l = 3$ and $k > 2$ describes the Z_4 parafermion quantum Hall states. As a more nontrivial check on these results, we now turn to the calculation of the ground state degeneracy on surfaces of arbitrary genus.

B. Ground-state degeneracy for genus g

The ground-state degeneracy on a genus g surface of the Z_4 parafermion quantum Hall state at filling fraction $\nu = \frac{2}{2k-3}$, where $k \ge 2$, is given by[24]

$$(k - 3/2)^g 2^{g-1}[(3^g + 1) + (2^{2g} - 1)(3^{g-1} + 1)]. \quad (33)$$

Note that the second factor, $2^{g-1}[(3^g + 1) + (2^{2g} - 1)(3^{g-1} + 1)]$, is the dimension of the space of conformal blocks on a genus g surface in the Z_4 parafermion CFT [see Eq. (C2)]. The degeneracy for the corresponding quantum Hall state is $(k - 3/2)^g = \nu^{-g}$ times this factor.

Let us consider the ground-state degeneracy on a genus g surface for the $U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory. Let $\{a_i\}$ and $\{b_i\}$, with $i = 1, \ldots, g$ be a basis for the homology cycles (see Fig. 1). The a_i (b_i) do not intersect each other, while a_i and b_j intersect if $i = j$. That is, the a_i and b_i form a canonical homology basis. There can be a Z_2 twist along any combination of these noncontractible loops. Thus there are 2^{2g} different sectors; one of them is untwisted while the other $2^{2g} - 1$ sectors are twisted. Let us first analyze the untwisted sector.
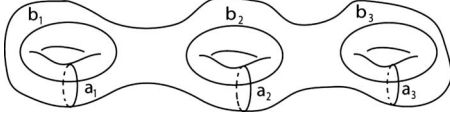
FIG. 1. Canonical homology basis for Σ_g.

It is known that the $(k,k,k-l)$ bilayer FQH states, which are described by the $U(1) \times U(1)$ Chern-Simons theory of Eq. (3) have a degeneracy of $|\det K|^g$, where the K matrix is

$$K = \begin{pmatrix} k & k-l \\ k-l & k \end{pmatrix}. \tag{34}$$

Thus the degeneracy for these bilayer states is $|l^g(2k-l)^g|$. These states may be written as

$$\otimes_i |n_i, m_i\rangle, \tag{35}$$

where the n_i and m_i are integers, $i = 1, \ldots, g$, and with the identifications [see Eq. (19)]

$$(n_i, m_i) \sim (n_i + k - l, m_i + k) \sim (n_i + k, m_i + k - l) \tag{36}$$

for each i. The action of the Z_2 on these states is to take

$$\otimes_i |n_i, m_i\rangle \rightarrow \otimes_i |m_i, n_i\rangle. \tag{37}$$

We must project onto the Z_2 invariant states. There are $|2k-l|^g$ diagonal states of the form $\otimes_i |n_i, n_i\rangle$. These are invariant under the Z_2. There are $|l^g(2k-l)^g| - |2k-l|^g$ off-diagonal states, and exactly half of them are Z_2 invariant. This gives a total of

$$(|l|^g + 1)|2k-l|^g/2 = (|l|^g + 1)|k - l/2|^g 2^{g-1} \tag{38}$$

different states, which for $l=3$ and $k \geq 1$ corresponds to the first term of Eq. (33).

Now consider the twisted sectors. To begin, suppose that there is a Z_2 twist along the a_g cycle, and no twists along any of the other cycles. Let Σ_g refer to the genus g surface. Let us consider the double cover $\hat{\Sigma}_{2g-1}$ of Σ_g, which is a genus $2g-1$ surface. It can be constructed as follows. Take two copies of Σ_g, referred to as Σ_g^1 and Σ_g^2, and cut both of them along their a_g cycle. Gluing them together in such a way that each end of the cut on one copy lands on the opposite end of the cut on the other copy leaves the $2g-1$ surface $\hat{\Sigma}_{2g-1}$ (see Fig. 2). The sheet exchange R is a map from $\hat{\Sigma}_{2g-1}$ to itself that satisfies $R \circ R = 1$ and which takes $\Sigma_g^1 \rightarrow \Sigma_g^2$ and vice versa.[25] We can now define a new, continuous gauge field c on $\hat{\Sigma}_{2g-1}$ as follows:

$$c(p) = \begin{cases} a(p), & p \in \Sigma_g^1 \\ \tilde{a}[R(p)], & p \in \Sigma_g^2. \end{cases} \tag{39}$$

Notice that because the gauge transformations get twisted also, c now behaves exactly as a typical $U(1)$ gauge field on a genus $2g-1$ surface. In particular, there are large gauge transformations which change the value of $\oint_{\alpha_i} c \cdot dl$ or $\oint_{\beta_i} c \cdot dl$ by 2π.
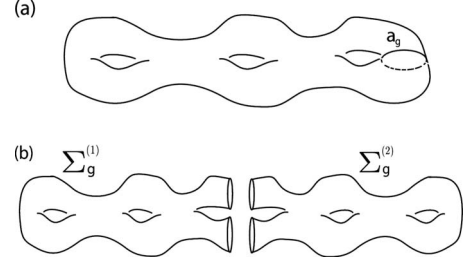
In terms of c, action (3) becomes

FIG. 2. (a) A single twist along the a_g direction. (b) Take two copies of Σ_g, cut them along the a_g cycle, and glue them together as shown. This yields a genus $2g-1$ surface. For the case $g=3$, we see explicitly that a genus 5 surface is obtained.

$$L = \int_{\hat{\Sigma}_{2g-1}} \frac{k}{4\pi} c(p) \partial c(p) + \frac{k-l}{4\pi} c(p) \partial c[R(p)]. \tag{40}$$

In terms of c, the Lagrangian is nonlocal; however this poses no difficulty. Fixing the gauge $c_0 = 0$, the equation of motion for c_0 is a constraint that enforces c to have zero field strength; that is, c is a flat connection.

Let $\{\alpha_i\}$ and $\{\beta_i\}$ be a basis of canonical homology cycles on $\hat{\Sigma}_{2g-1}$, with $i = 1, \ldots, 2g-1$. We can choose α_i and β_i in such a way that the sheet exchange R acts on these cycles as follows:

$$R\alpha_i = \alpha_{i+g-1}, \quad R\beta_i = \beta_{i+g-1},$$

$$R\alpha_{2g-1} = \alpha_{2g-1}, \quad R\beta_{2g-1} = \beta_{2g-1}, \tag{41}$$

where $i = 1, \ldots, g-1$. The dual basis is the set of one forms ω_i and η_i, which satisfy

$$\int_{\alpha_i} \omega_j = \delta_{ij}, \quad \int_{\beta_i} \omega_j = 0,$$

$$\int_{\alpha_i} \eta_j = 0, \quad \int_{\beta_i} \eta_j = \delta_{ij}. \tag{42}$$

Since c must be a flat connection, we can parametrize it as

$$c = c_1 dx^1 + c_2 dx^2 = 2\pi(x^i \omega_i + y^i \eta_i). \tag{43}$$

Two connections c and c' are gauge equivalent if

$$x'^i - x^i = \text{integer}, \quad y'^i - y^i = \text{integer}. \tag{44}$$

Furthermore, from the definition of c [Eq. (39)], we see that the Z_2 action is the same as the action of the sheet exchange R,

$$(x^i, y^i) \rightarrow (x^{R(i)}, y^{R(i)}), \tag{45}$$

where

$$R(i) = \begin{cases} i + g - 1 & \text{for } i = 1, \ldots, g-1 \\ i - g + 1 & \text{for } i = g, \ldots, 2g-2 \\ 2g - 1 & \text{for } i = 2g-1. \end{cases} \tag{46}$$

Substituting into action (40) and using the fact that $\int_{\hat{\Sigma}_{2g-1}} \omega_j \wedge \eta_k = \delta_{jk}$ and $\int_{\hat{\Sigma}_{2g-1}} \omega_j \wedge \omega_k = \int_{\hat{\Sigma}_{2g-1}} \eta_j \wedge \eta_k = 0$, we obtain
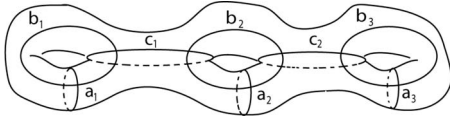
FIG. 3. Canonical homology basis for Σ_g.

$$L = 2\pi k y^i \dot{x}^i + 2\pi(k-l) y^i \dot{x}^{R(i)}. \tag{47}$$

Apart from the variables with $i = 2g-1$, this action looks like the action for a bilayer $(k,k,k-l)$ state on a genus $g-1$ surface. Therefore, we can easily deduce that quantizing this system before imposing the invariance under the Z_2 action gives $|l|^{g-1}|2k-l|^{g-1} \times |2k-l|$ different states. The extra factor $|2k-l|$ comes from the variables with $i = 2g-1$, which independently behave as the zero modes of a $U(1)_{2k-l}$ CS theory on a torus. We can write the states as

$$|n_{2g-1}\rangle \otimes_i |n_i, n_{R(i)}\rangle, \tag{48}$$

for $i = 1, \ldots, g-1$ and with the identifications

$$n_{2g-1} \sim n_{2g-1} + 2k - l, \tag{49}$$

$$(n_i, n_{R(i)}) \sim (n_i + k, n_{R(i)} + k - l) \sim (n_i + k - l, n_{R(i)} + k). \tag{50}$$

Note the n_i are all integer. Now we must project onto the Z_2 invariant sector. The action of the Z_2 is to take

$$|n_{2g-1}\rangle \otimes_i |n_i, n_{R(i)}\rangle \rightarrow |n_{2g-1}\rangle \otimes_i |n_{R(i)}, n_i\rangle. \tag{51}$$

Suppose $n_i = n_{R(i)}$ for each i. Such states are already Z_2 invariant; there are $|2k-l| \times |2k-l|^{g-1}$ of them. The remaining states for which $n_i \neq n_{R(i)}$ for at least one i always change under the Z_2 action. The Z_2 invariant combination is

$$|n_{2g-1}\rangle \otimes_i (|n_i, n_{R(i)}\rangle + |n_{R(i)}, n_i\rangle). \tag{52}$$

There are $|2k-l| \times \frac{|l|^{g-1}|2k-l|^{g-1} - |2k-l|^{g-1}}{2}$ of these. In total there-fore there are

$$|2k-l|^g \frac{|l|^{g-1} + 1}{2} = |k - l/2|^g (|l|^{g-1} + 1) 2^{g-1} \tag{53}$$

states in this particular twisted sector.

Now it turns out that each of the $2^{2g} - 1$ twisted sectors (which generically has many Z_2 twists along many different noncontractible loops) yield the same number of ground states as the sector in which there is a single twist along just the a_g cycle. One can understand this by considering the modular group, or mapping class group, of Σ_g. This is the group of diffeomorphisms on Σ_g modulo those that are continuously connected to the identity. They are generated by "Dehn twists," which correspond to cutting the surface along some noncontractible loop, rotating one side by 2π, and gluing the two sides back together. The mapping class group of Σ_g can be generated by Dehn twists along the loops a_i, b_i, and c_i, shown in Fig. 3. Elements of the mapping class group are symmetries of the topological field theory, which means that they are represented by unitary operators on the quantum Hilbert space. In particular, the dimension of the space of states for a given twisted sector is equivalent to that of a

different twisted sector if they can be related by the action of an element of the mapping class group. In the following we sketch how, using Dehn twists, one can go from any arbitrary twisted sector to the sector in which there is a single Z_2 twist along only the a_g cycle.

First note that a Z_2 twist along some cycle γ is equivalent to having a Z_2 twist along $-\gamma$, and that a Z_2 twist along $\gamma + \gamma$ is equivalent to having no Z_2 twist at all. Since we are here concerned only with the properties of the Z_2 twists, we use these properties in the algebra below. In other words, the algebra below will be defined over Z_2 because we are only concerned with Z_2 twists along various cycles.

Let us call A_i, B_i, and C_i the Dehn twists that act along the a_i, b_i, and c_i cycles. Notice that a Z_2 twist along a_i and a_{i+1} is equivalent to a Z_2 twist along c_i. Let us consider the action of A_i, B_i, and C_i on Z_2 twists along the a_i and b_i cycles,

$$A_i : a_i \rightarrow a_i$$

$$b_i \rightarrow a_i + b_i,$$

$$B_i : a_i \rightarrow a_i + b_i$$

$$b_i \rightarrow b_i,$$

$$C_i : a_i \rightarrow a_i$$

$$b_i \rightarrow b_i + c_i = b_i + a_i + a_{i+1}$$

$$a_{i+1} \rightarrow a_{i+1}$$

$$b_{i+1} \rightarrow b_{i+1} + c_i = b_{i+1} + a_i + a_{i+1}. \tag{54}$$

Z_2 twists along all other cycles are left unchanged. Notice in particular that $A_i^{-1} B_i : a_i \rightarrow b_i$ so that a Z_2 twist along a_i is equivalent to one along $a_i + b_i$, which is also equivalent to one along b_i. As a result, we can see that the configuration of Z_2 twists can be labeled only by considering which of the g handles have any twists at all. Furthermore, since we can rearrange the holes without changing the topology, the configuration of Z_2 twists is actually labeled by considering how many of the g handles have twists.

Suppose that two of the g handles have Z_2 twists. Since we have freedom to rearrange the holes, we can consider the situation in which two neighboring handles each have a Z_2 twist. Since twists along a_i, $a_i + b_i$, and b_i are all equivalent, let us suppose that one handle has a twist along its b cycle, while the other handle has a twist along its a cycle. That is, we are considering the situation in which there is a twist along $b_i + a_{i+1}$. Now, performing the Dehn twist C_i, we have

$$C_i : b_i + a_{i+1} \rightarrow b_i + a_i + a_{i+1} + a_{i+1} = b_i + a_i. \tag{55}$$

Thus we see that the case with Z_2 twists for two handles is equivalent to that for a Z_2 twist along a single handle. From this, it follows that the case with n handles having Z_2 twists is equivalent to the case where only a single handle has a Z_2 twist.

Therefore, under actions of the Dehn twists, any arbitrary twisted sector goes into the sector in which there is a single

twist along the a_g cycle. This means that the dimension of the Hilbert space is the same for each of the $(2^{2g}-1)$ twisted sectors and in particular is equal to that for the sector in which there is a single twist along a_g. We computed that situation explicitly [see Eq. (53)], so we can conclude that the number of ground states on a genus g surface for the $U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory is

$$S_g(k,l) = |k - l/2|^g 2^{g-1} [(|l|^g + 1) + (2^{2g}-1)(|l|^{g-1}+1)]. \tag{56}$$

For $l=3$, this corresponds to the degeneracy of the Z_4 parafermion quantum Hall state that we expect from a CFT calculation [see Eq. (33)]. When $l=2$, we get

$$S_g(k,2) = |k-1|^g [2^{g-1}(2^g+1)]^2, \tag{57}$$

which corresponds to the degeneracy of the $\nu = \frac{1}{k-1}$ Pfaffian quantum Hall state times an extra factor of $2^{g-1}(2^g+1)$, which is the dimension of the space of conformal blocks of the Ising CFT on a genus g surface. This again confirms the notion that for $l=2$, this theory corresponds to the Pfaffian state with an extra copy of the Ising model.

IV. QUANTUM DIMENSIONS OF QUASIPARTICLES FROM GROUND-STATE DEGENERACY

In the last section we found the ground-state degeneracy, S_g, of the $U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory on a surface of genus g. From S_g we can deduce some topological properties of the quasiparticles. It is well known, for example, that S_1, the ground-state degeneracy on a torus, is equal to the number of topologically distinct quasiparticles. Here we show that from S_g we can also obtain the quantum dimensions of each of the quasiparticles.

The quantum dimension d_γ of a quasiparticle denoted by γ has the following meaning. For n quasiparticles of type γ at fixed positions, the dimension of the Hilbert space grows as d_γ^n. For Abelian quasiparticles at fixed positions, there is no degeneracy of states, so the quantum dimension of an Abelian quasiparticle is 1. The quantum dimension d_γ can be obtained from the fusion rules of the quasiparticles, $N_{\gamma\gamma'}^{\gamma''}$: d_γ is the largest eigenvalue of the fusion matrix N_γ, where $(N_\gamma)_{\gamma'}^{\gamma''} = N_{\gamma\gamma'}^{\gamma''}$. From the quantum dimensions d_γ, we can obtain S_g through the formula,[24,26]

$$S_g = D^{2(g-1)} \sum_{\gamma=0}^{N-1} d_\gamma^{-2(g-1)}, \tag{58}$$

where N is the number of quasiparticles, d_γ is the quantum dimension of quasiparticle γ, and $D = \sqrt{\sum_\gamma d_\gamma^2}$ is the "total quantum dimension." Remarkably, this formula also implies that if we know S_g for any g, then we can uniquely determine all of the quantum dimensions d_γ. To see how, let us first order the quasiparticles so that $d_{\gamma+1} \geq d_\gamma$. Notice that the identity has unit quantum dimension, $d_0 = 1$, and suppose that $d_i = 1$ for $i = 0, \ldots, i_0$ ($i_0 \geq 0$), $d_{i_0+1} > 1$. Now consider

$$\lim_{g\to\infty} \frac{S_{g+1}}{S_g} = D^2 \lim_{g\to\infty} \frac{i_0 + \sum\limits_{\gamma=i_0+1}^{N-1} d_\gamma^{-2g}}{i_0 + \sum\limits_{\gamma=i_0+1}^{N-1} d_\gamma^{-2(g-1)}} = D^2. \tag{59}$$

We see that the total quantum dimension D can be found by computing $\lim_{g\to\infty} \frac{S_{g+1}}{S_g}$. Now define

$$\widetilde{S}_g^{(1)} \equiv \frac{S_g}{D^{2(g-1)}} - 1 = \sum_{\gamma=1}^{N-1} d_\gamma^{-2(g-1)} \tag{60}$$

and suppose that $d_1, \ldots, d_{i_1}$ all have the same quantum dimension. Now consider the following limit:

$$\lim_{g\to\infty} \frac{\widetilde{S}_{g+1}^{(1)}}{\widetilde{S}_g^{(1)}} = \lim_{g\to\infty} \frac{d_1^{-2g}\left(i_1 + \sum\limits_{\gamma=i_1+1}^{N-1} d_\gamma^{-2g}\right)}{d_1^{-2(g-1)}\left(i_1 + \sum\limits_{\gamma=i_1+1}^{N-1} d_\gamma^{-2(g-1)}\right)} = d_1^{-2}. \tag{61}$$

We see that d_1 can be determined by computing $\lim_{g\to\infty} \frac{\widetilde{S}_{g+1}^{(1)}}{\widetilde{S}_g^{(1)}}$. This allows one to define

$$\widetilde{S}_g^{(2)} \equiv \widetilde{S}_g^{(1)} - d_1^{-2(g-1)} = \sum_{\gamma=2}^{N-1} d_\gamma^{-2(g-1)}, \tag{62}$$

and in turn we find $d_2^{-2} = \lim_{g\to\infty} \frac{\widetilde{S}_{g+1}^{(2)}}{\widetilde{S}_g^{(2)}}$. Proceeding in this way, one can obtain d_i, then define

$$\widetilde{S}_g^{(i+1)} \equiv \widetilde{S}_g^{(i)} - d_i^{-2(g-1)} = \sum_{\gamma=i+1}^{N-1} d_\gamma^{-2(g-1)}, \tag{63}$$

and then compute d_{i+1} from $\widetilde{S}_g^{(i+1)}$,

$$d_{i+1}^{-2} = \lim_{g\to\infty} \frac{\widetilde{S}_{g+1}^{(i+1)}}{\widetilde{S}_g^{(i+1)}}. \tag{64}$$

Thus we can see that in this way all of the quantum dimensions of the quasiparticles can be obtained from the formula for the ground-state degeneracy on a genus g surface.

Carrying out this procedure for the $U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory, we find that when $|l| < 4$, the quantum dimensions of the quasiparticles take one of three different values. $2|2k-l|$ of them have quantum dimension 1, $2|2k-l|$ of them have quantum dimension $\sqrt{l}$, and the remaining $(|l|-1)|2k-l|/2$ of them have quantum dimension 2. The total quantum dimension, for all (k,l), is

$$D^2 = 4|l(2k-l)|. \tag{65}$$

For $l=3$ this coincides exactly with the quantum dimensions of the quasiparticles in the $\nu = \frac{2}{2k-3}$ Z_4 parafermion FQH states.

V. QUASIPARTICLES

When we refer to quasiparticles in a Chern-Simons theory, we are referring to topological defects in the configuration of the gauge fields. For instance, for a Chern-Simons theory at level k with a simple Lie group G, a quasiparticle is represented by a unit of flux in an integrable representation of the affine Lie algebra $\hat{g}_k$, where g is the Lie algebra of G. The partition function of the Chern-Simons theory in the presence of external sources of quasiparticles is

$$Z(\{C_i, R_i\}) = \int \mathcal{D}A \prod_i W_{R_i}(C_i) e^{iS_{CS}[A]}, \qquad (66)$$

where the Wilson loop operator $W_R(C)$ is defined as

$$W_R(C) = \mathrm{Tr}_R \mathcal{P} \exp\left[i\oint_C A \cdot dl \right]. \qquad (67)$$

Tr_R is a trace in the representation R, $\mathcal{P}$ refers to path ordering, and C is a loop describing the world line of the quasiparticle. Furthermore, the action of the quantum operator $\hat{W}_{R_i}(C)$ is to take one ground state to another when C is a noncontractible loop in space.

In the $U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory, there are several types of quasiparticles to consider. Some of the quasiparticles are related to the Wilson loop operators for the $U(1)$ gauge fields; some are neutral under the Z_2 gauge field while others carry Z_2 charge. There are also Z_2 vortices, which we explicitly analyze in the following section.

A. Z_2 vortices

One basic excitation in a theory with a Z_2 gauge symmetry is a Z_2 vortex. In the context of $U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory, a Z_2 vortex is, roughly speaking, a point around which the $U(1)$ gauge fields transform into each other. Here we compute the degeneracy of states in the presence of n pairs of Z_2 vortices at fixed positions; we find that this degeneracy grows like $|l|^n$, and therefore the Z_2 vortices can be identified with the non-Abelian quasiparticles with quantum dimension $\sqrt{|l|}$. We can in fact obtain the formula for the degeneracy more precisely and find that it agrees exactly, for $l=3$, with results from the Z_4 parafermion FQH states.

The basic idea is that a sphere with n pairs of Z_2 vortices can be related to a $U(1)_l$ Chern-Simons theory on a genus $g=n-1$ Riemann surface. We will find that the Z_2 invariant subspace of this theory has $(|l|^{n-1}+1)/2$ states while the Z_2 noninvariant subspace has $(|l|^{n-1}-1)/2$ states when l is odd.

We may define a pair of Z_2 vortices more precisely as a one-dimensional closed submanifold γ of our spatial two-manifold M_0. The two boundary points of γ are thought of as the location of the Z_2 vortices. The gauge field A_μ is defined on $M=M_0 \backslash \gamma$, with the following boundary conditions along γ:
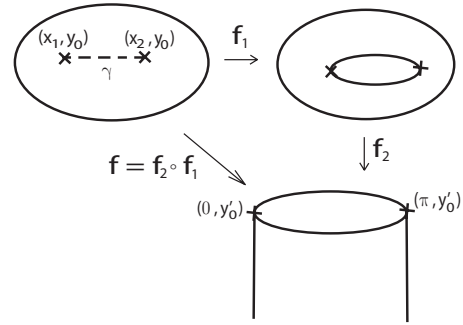

FIG. 4. Consider the diffeomorphism f, which takes the neighborhood of a pair of Z_2 vortices to the end of a cylinder. We can imagine f as a composition of two maps, the first which expands the cut γ to a hole, and a second one which maps the result to the end of a cylinder.

$$\lim_{p \to p_0^\pm} A_\mu(p) = \lim_{p \to p_0^\mp} \sigma_1 A_\mu(p) \sigma_1 \qquad (68)$$

for every point $p_0 \in \gamma$. The limit $p \to p_0^{+(-)}$ means that the limit is taken approaching one particular side (or the other) of γ.

Consider the action of a diffeomorphism $f: M \to M$, which takes $p \to p' = f(p)$. The Chern-Simons action is a topological invariant and is therefore invariant under diffeomorphisms. However, the gauge fields transform along with the coordinates, which means that the boundary conditions at the boundary of $M = M_0 \backslash \gamma$ will change. Let us determine how the boundary conditions on A change under the action of the diffeomorphism f, which acts in the way indicated in Fig. 4 in the neighborhood of a pair of Z_2 vortices connected by γ.

Choosing a coordinate chart in the neighborhood of a pair of Z_2 vortices, we can write the action of f as

$$x^\mu \to x'^\mu,$$

$$a_\mu \to a'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} a_\nu. \qquad (69)$$

Let us choose the coordinates x^μ such that (see Fig. 4)

$$\gamma = \{(x, y_0) | x_1 \leq x \leq x_2\}. \qquad (70)$$

The two Z_2 vortices are located at the two ends of γ and f maps the neighborhood of these Z_2 vortices to the end of a cylinder; the boundary M in this neighborhood gets mapped to a circle. In terms of the new coordinates x'^μ, this neighborhood of M gets mapped to

$$\{(x', y') | y' < y_0', x' \in \mathbb{R}\%2\pi\}. \qquad (71)$$

The location of the Z_2 vortices in the new coordinates is taken to be at $(0, y_0')$ and (π, y_0'). Fix some small $\epsilon > 0$. Let us choose an f that takes

$$(x_0, y_0 \pm \epsilon) \to (\mp x_0', y_0' - \epsilon) \qquad (72)$$

for $x_1 < x_0 < x_2$. It is easy to see that as ϵ is taken to zero, we have
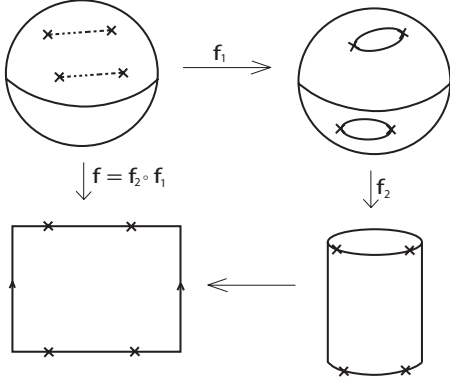
FIG. 5. Two pairs of Z_2 vortices on a sphere. This sequence of diffeomorphisms illustrates that this situation is equivalent to M being a cylinder.

$$\lim_{\epsilon \to 0^+} \left. \frac{\partial x'^i}{\partial x^j} \right|_{(x_0, y_0 \pm \epsilon)} = \mp \delta_j^i. \tag{73}$$

Applying Eq. (69), we can immediately see that the boundary conditions for A_μ' acquire an additional minus sign,

$$A_\mu'(\pm x', y_0') = -\sigma_1 A_\mu'(\mp x', y_0')\sigma_1. \tag{74}$$

Let us now study the cases $n=1$ and $n=2$ for $M_0=S^2$ before attempting to generalize to arbitrary n. We begin by considering the case $n=2$, the case of two pairs of Z_2 vortices on a sphere. Consider also the diffeomorphism f shown in Fig. 5. Clearly, the situation with two pairs of Z_2 vortices on a sphere is equivalent to having the gauge field A_μ defined on the space

$$M = \{(x, y) | 0 \le y \le L, x \in \mathbb{R}\%L\} \tag{75}$$

for any L with the following periodicity/boundary conditions:

$$A_\mu(x + L, y) = A_\mu(x, y),$$

$$A_\mu(x, L) = -\sigma_1 A_\mu(-x, L)\sigma_1,$$

$$A_\mu(x, 0) = -\sigma_1 A_\mu(-x, 0)\sigma_1, \tag{76}$$

and with the action of Eq. (3). We can now define a new continuous field c_μ defined on

$$\widetilde{M} = \{(x, y) | x \in \mathbb{R}\%L, y \in \mathbb{R}\%2L\} \tag{77}$$

as follows:

$$c_\mu(x, y) = \begin{cases} a_\mu(x, y), & 0 \le y \le L \\ -\widetilde{a}_\mu(-x, 2L - y), & L \le y \le 2L, \end{cases} \tag{78}$$

where now c_μ is doubly periodic,

$$c_\mu(x, y) = c_\mu(x + L, y) = c_\mu(x, y + 2L). \tag{79}$$

Recall that the $U(1) \times U(1)$ gauge transformations on A_μ are of the form

$$U = \begin{pmatrix} e^{if} & 0 \\ 0 & e^{ig} \end{pmatrix},$$

$$A_\mu \to A_\mu + iU\partial_\mu U^{-1}. \tag{80}$$

These gauge transformations must preserve the boundary conditions (76) on A_μ. This implies that U obeys the following boundary conditions:

$$U(x + L, y) = U(x, y),$$

$$U(x, L) = \sigma_1 U^{-1}(-x, L)\sigma_1,$$

$$U(x, 0) = \sigma_1 U^{-1}(-x, 0)\sigma_1. \tag{81}$$

Just as we defined c_μ from A_μ, we can define the gauge transformation that acts on c_μ in the following way:

$$h(x, y) = \begin{cases} f(x, y), & 0 \le y \le L \\ -g(-x, 2L - y), & L \le y \le 2L \end{cases} \tag{82}$$

so that the gauge transformation U acts on c_μ as

$$c_\mu \to c_\mu - \partial_\mu h. \tag{83}$$

So we see that c_μ behaves like a typical $U(1)$ gauge field defined on a torus. In particular, the only condition on $h(x, y)$ is that $e^{ih(x,y)}$ be doubly periodic, which allows for the possibility of large gauge transformations along the two noncontractible loops of the torus.

In the $A_0 = 0$ gauge, the Lagrangian can be written as

$$L = \epsilon^{ji} \int d^2x \left[\frac{k}{4\pi}(a_i \dot{a}_j + \widetilde{a}_i \dot{\widetilde{a}}_j) + \frac{k-l}{4\pi}(a_i \dot{\widetilde{a}}_j + \widetilde{a}_i \dot{a}_j) \right], \tag{84}$$

where the integration is over the region $0 \le x, y \le L$. In terms of c_μ,

$$\int_0^L dx \int_0^L dy (a_i \dot{a}_j + \widetilde{a}_i \dot{\widetilde{a}}_j) = \int_0^L dx \int_0^{2L} dy c_i \dot{c}_j. \tag{85}$$

Using $\widetilde{a}_j(x, y) = -c_j(-x, 2L - y)$, we see

$$\int_0^L \int_0^L d^2x a_i \dot{\widetilde{a}}_j = -\int_0^L \int_0^L d^2x c_i(x, y)\dot{c}_j(-x, 2L - y),$$

$$\int_0^L \int_0^L d^2x \widetilde{a}_i \dot{a}_j = -\int_0^L dx \int_L^{2L} dy c_i(x, y)\dot{c}_j(-x, 2L - y). \tag{86}$$

Therefore we can write the action in terms of c_μ as

$$L = \epsilon^{ji} \int_0^L dx \int_0^{2L} dy \left[\frac{k}{4\pi}c_i \dot{c}_j - \frac{k-l}{4\pi}c_i(x, y)\dot{c}_j(-x, 2L - y) \right]. \tag{87}$$

The equation of motion for c_0 serves as a constraint for zero field strength, which implies that we can parameterize c_i as

$$c_i(x, y, t) = \frac{2\pi}{L_i}X_i(t) + \widetilde{c}_i(x, y, t). \tag{88}$$

The large gauge transformations take $X_i \to X_i + $ integer. The topological degeneracy is given by the degeneracy of this
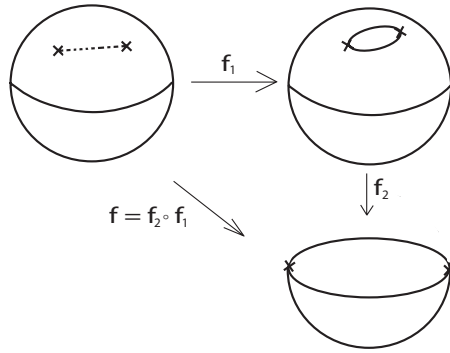
FIG. 6. A single pair of Z_2 vortices on a sphere. This sequence of diffeomorphisms illustrates that this situation is equivalent to M being a hemisphere, but with a different set of boundary conditions on A_μ.

zero-mode sector. The action of the zero-mode sector is found upon substituting Eq. (88) into action (87),

$$L = 2\pi l X_2 \dot{X}_1. \qquad (89)$$

Now we must make sure that we project onto the Z_2 invariant sector. The Z_2 exchanges a and $\tilde{a}$, so it takes $c(x,y) \rightarrow -c(x, y+L)$ if $y \leq L$ and $c(x,y) \rightarrow -c(x, y-L)$ if $y \geq L$. Thus, the action of the Z_2 is to take the zero modes to minus themselves: $X_i \rightarrow -X_i$. The states can be labeled by $|n\rangle$, where n is an integer and with the identifications $|n\rangle = |n+l\rangle$. Thus, before the projection, there are $|l|$ states. If l is even, then there are two fixed points of the of the Z_2 action, so in all there are $|l|/2 + 1$ Z_2 invariant states. If l is odd, there are only $(|l|+1)/2$ Z_2 invariant states.

Consider now the case of a single pair of Z_2 vortices on a sphere and the diffeomorphism f shown in Fig. 6. Clearly, the situation with a single pair of Z_2 vortices is equivalent to having the gauge field A_μ defined on a hemisphere, but with modified boundary conditions on the A_μ. Let the angular coordinates (θ, φ) be defined so that the locations of the two Z_2 vortices are $(\pi/2, 0)$ and $(\pi/2, \pi)$ for the left and right vortices, respectively. The south pole is at $\theta = \pi$. As in the previous case with two Z_2 vortices, the boundary conditions on A_μ at $\theta = \pi/2$ are as follows:

$$A_\mu(\pi/2, \varphi) = -\sigma_1 A_\mu(\pi/2, -\varphi)\sigma_1. \qquad (90)$$

As a result, we can define a new continuous gauge field c_μ on a sphere as follows:

$$c_\mu(\theta, \varphi) = \begin{cases} a_\mu(\theta, \varphi), & \pi/2 \leq \theta \leq \pi \\ -\tilde{a}_\mu(\pi - \theta, -\varphi), & 0 \leq \theta \leq \pi/2. \end{cases} \qquad (91)$$

It is easy to see that in this case, there is no possibility for large gauge transformations or holonomies around noncontractible loops. The Lagrangian will be given by an expression similar to Eq. (87), but this time the degeneracy will be 1.

We now tackle the case for general n. Suppose that there are n pairs of Z_2 vortices on a sphere. We will define the new gauge field c_μ on a genus $g = (n-1)$ surface in the following way. From Fig. 7, we can clearly see that the situation with four pairs of vortices is equivalent to having a gauge
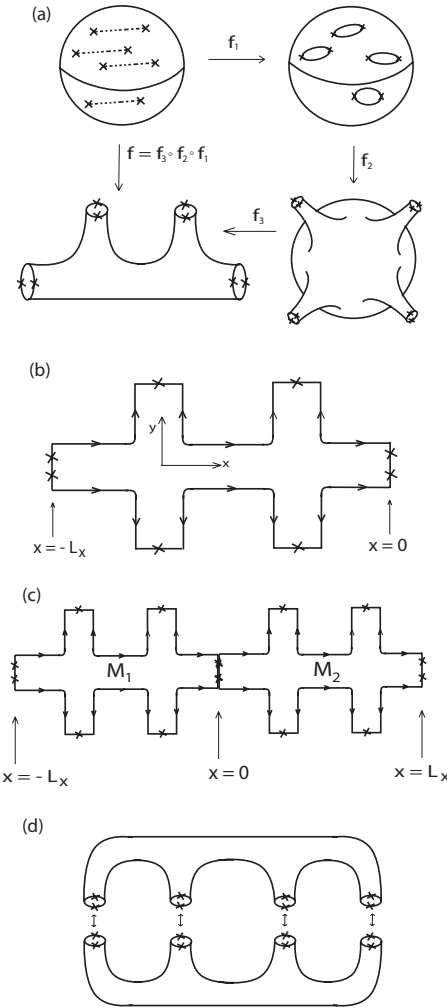

FIG. 7. Four pairs of Z_2 vortices on a sphere. (a) This sequence of diffeomorphisms shows that we can think of the situation with this many vortices as a gauge field defined on the surface shown in the lower left figure, which looks like half of a genus $g=3$ surface. (b) The figure in the lower left of (a) can be cut open as shown here. The arrows on the figure indicate how the points on the boundaries should be identified. (c) Two copies of the figure in (b). (d) Gluing together two copies along their boundaries gives a genus $g=3$ surface. For general n, this procedure gives a surface of genus $g=n-1$.

field A_μ defined on the surface shown in the lower left of Figs. 7(a) and 7(b) with modified boundary conditions. The generalization from four to n is obvious. Consider the space shown in Fig. 7(c), which contains two copies of the original space. Parametrize this doubled space with the coordinates $\vec{r} = (x, y)$. We will refer to the copy on the left side, which has $x \leq 0$, as M_1; the copy on the right side, which has $x \geq 0$, will be referred to as M_2. Suppose that the length in the x direction of each copy is L_x so that the total horizontal length of the doubled space is $2L_x$.

Consider a map R defined on this doubled space with the following properties: R takes M_1 to M_2 and M_2 into M_1 in such a way that $R \circ R = 1$, it has unit Jacobian, and it maps the boundaries of M_1 and M_2 into each other. The way it maps ∂M_1 and ∂M_2 into each other is illustrated in Fig. 7(d); if we

identify ∂M_1 and ∂M_2 using the map R, then we obtain a surface of genus $g = n-1$, which we call M. In the coordinates illustrated in Fig. 7(c), this way of mapping ∂M_1 and ∂M_2 results in the following boundary conditions on A_μ:

$$A_\mu(x,y) = -\sigma_1 A_\mu[R^1(x,y) - L_x, R^2(x,y)]\sigma_1 \quad (92)$$

for $(x,y) \in \partial M_1$ and where $R^i(x,y)$ is the ith coordinate of R [note that $A_\mu(x,y)$ is only defined for $-L_x \leq x \leq 0$]. This allows us to define a continuous gauge field c_μ, defined on the doubled space M, in the following way:

$$c_\mu(x,y) = \begin{cases} a_\mu(x,y), & x \leq 0 \\ -\tilde{a}_\mu[R(x,y)], & x \geq 0. \end{cases} \quad (93)$$

We now rewrite the various terms in the action in terms of c_μ,

$$\int_{M_1} d^2x\,\tilde{a}_i\dot{\tilde{a}}_j = \int_{M_2} d^2x\,\tilde{a}_i[R(x,y)]\dot{\tilde{a}}_j[R(x,y)] = \int_{M_2} d^2x\,c_i\dot{c}_j. \quad (94)$$

The cross terms give a nonlocal term in the action,

$$\int_{M_1} d^2x\,a_i\dot{\tilde{a}}_j = -\int_{M_1} d^2x\,c_i(x,y)\dot{c}_j[R(x,y)],$$

$$\int_{M_1} d^2x\,\tilde{a}_i\dot{a}_j = -\int_{M_2} d^2x\,c_i(x,y)\dot{c}_j[R(x,y)]. \quad (95)$$

Thus the Lagrangian is

$$L = \epsilon^{ji}\int_M d^2x\left[\frac{k}{4\pi}c_i\dot{c}_j - \frac{k-l}{4\pi}c_i(x,y)\dot{c}_j[R(x,y)]\right]. \quad (96)$$

As usual in pure Chern-Simons theory, the equation of motion for c_0 implies that the gauge field must be flat. It is therefore characterized by the value of $\oint_C c\cdot dl$ along its noncontractible loops. To parametrize the gauge field, as is typical we introduce a canonical homology basis α_i and β_i such that α_i (β_i) do not intersect while α_i and β_j intersect if $i=j$. Then we introduce the dual basis ω_i and η_j, which satisfy

$$\int_{\alpha_i}\omega_j = \delta_{ij}, \quad \int_{\beta_i}\omega_j = 0,$$

$$\int_{\alpha_i}\eta_j = 0, \quad \int_{\beta_i}\eta_j = \delta_{ij}. \quad (97)$$

Since c must be a flat connection, we can parametrize it as

$$c = c_1 dx^1 + c_2 dx^2 = 2\pi(x^i\omega_i + y^i\eta_i). \quad (98)$$

Two connections c and c' are gauge equivalent if

$$x'^i - x^i = \text{integer}, \quad y'^i - y^i = \text{integer}. \quad (99)$$

Notice that here the action of R is trivial on the canonical homology cycles. This is because of the way the genus $n-1$ surface was glued together from its pieces [see Fig. 7(d)]. This is in contrast to Eq. (41), which we obtained when we were analyzing the ground-state degeneracy on higher genus

surfaces. Therefore, the action in terms of the x^i and y^i becomes simply

$$L = 2\pi l x^i \dot{y}^i \quad (100)$$

for $i = 1, \ldots, n-1$. However, the Z_2 action here is not exactly the same as the action of the sheet exchange map R. This is because the Z_2 exchanges a and $\tilde{a}$, so it takes $c(x,y) \to -c[R(x,y)]$. Thus, the action of the Z_2 is to change the sign of the x^i and y^i: $x^i \to -x^i$ and $y^i \to -y^i$ for every i, under the action of the Z_2. Before projection, it is clear that we have $|l|^{n-1}$ states. These can be labeled in the following way:

$$\otimes_{i=1}^{n-1}|m_i\rangle, \quad (101)$$

where m_i is an integer and $m_i \sim m_i + l$. The Z_2 action takes $m_i \to -m_i$. So if l is odd, there is one state that is already Z_2 invariant: the state with $m_i = 0$ for all i. There are $|l|^{n-1} - 1$ remaining states, and exactly half of them are Z_2 invariant. Thus if l is odd, the degeneracy of (Z_2-invariant) states in the presence of n pairs of Z_2 vortices on a sphere is $(|l|^{n-1} + 1)/2$. For $l=2$, $|m_i\rangle = |0\rangle$ or $|1\rangle$, which are both Z_2 invariant, so for $l=2$ the degeneracy in the presence of n pairs of Z_2 vortices on a sphere is 2^{n-1}. One may ask also about the number of states that are not Z_2 invariant. These may correspond to a different set of quasiparticle states that carry Z_2 charge. We see that there are $(3^{n-1}-1)/2$ Z_2 noninvariant states for $l=3$ if there are n pairs of Z_2 vortices on a sphere.

B. Comparison to quasiparticles in Z_4 parafermion and Pfaffian FQH states

Let us now compare the results from the previous section to the quasiparticles in the Pfaffian and Z_4 parafermion FQH states. The topological properties of the quasiparticles in FQH states can be computed through the pattern of zeros approach[24,27,28] or through their connection to conformal field theory.[4,14] In the Pfaffian quantum Hall state, there are two main types of quasiparticles, corresponding to two different representations of a magnetic translation algebra.[24] These two classes of quasiparticles are commonly labeled in the following way:

$$\psi e^{iQ(1/\sqrt{\nu})\varphi}, \quad \sigma e^{iQ(1/\sqrt{\nu})\varphi}, \quad (102)$$

where ψ is the Majorana fermion and σ is the spin field of the Ising CFT. Q is the charge of the quasiparticle and ν is the filling fraction of the quantum Hall state. The ones made of ψ are Abelian; there are $2q$ of them when the filling fraction is $\nu = 1/q$. The ones made of σ are non-Abelian; there are q of them and their quantum dimension is $\sqrt{2}$. In the presence of n pairs of the σ quasiparticles, the Pfaffian state has a degeneracy of 2^{n-1} on a sphere. This follows from the fusion rules of the conformal primary fields in the Ising CFT,

$$\psi\psi = 1,$$

$$\sigma\sigma = 1 + \psi,$$

$$\psi\sigma = \sigma. \quad (103)$$

Similarly, the quasiparticles of the Z_4 parafermion state compose three different representations of a magnetic trans-

TABLE I. Some values of the Z_2 vortex degeneracy for $l=3$ for the Z_2 invariant states, given by $(3^{n-1}+1)/2$, and for the Z_2 noninvariant states given by $(3^{n-1}-1)/2$.

No. Z_2 vortex pairs n	No. Z_2 inv. states $(3^{n-1}+1)/2$	No. Z_2 noninv. states $(3^{n-1}-1)/2$
1	1	0
2	2	1
3	5	4
4	14	13
5	41	40
6	122	121

TABLE II. Number of ways of fusing to the identity or to Φ_4^0 for the fusion of $(V_\sigma V_{\bar\sigma})^n$, where $V_\sigma = \Phi_1^1 e^{iQ(1/\sqrt{\nu})\varphi}$ and $V_{\bar\sigma} = \Phi_{-1}^1 e^{-iQ(1/\sqrt{\nu})\varphi}$.

n	No. of ways to fuse to $\Phi_0^0=1$	No. of ways to fuse to Φ_4^0
1	1	0
2	2	1
3	5	4
4	14	13
5	41	40
6	122	121

lation algebra, and these three classes are commonly labeled as

$$e^{iQ\sqrt{1/\nu}\varphi}, \quad \Phi_1^1 e^{iQ\sqrt{1/\nu}\varphi}, \quad \Phi_2^2 e^{iQ\sqrt{1/\nu}\varphi}. \quad (104)$$

When the filling fraction is $\nu = \frac{2}{2k-3}$, there are $2(2k-3)$ Abelian quasiparticles, $2(2k-3)$ of the Φ_1^1 quasiparticles, and $2k-3$ of the Φ_2^2 quasiparticles. The Φ_1^1 quasiparticles have quantum dimension $\sqrt{3}$ and the Φ_2^2 quasiparticles have quantum dimension 2.

The parafermionic primary fields in the Z_4 parafermion CFT have the fusion rules,

$$\Phi_m^0 \times \Phi_{m'}^l = \Phi_{m+m'}^l,$$

$$\Phi_1^1 \times \Phi_1^1 = \Phi_2^2 + \Phi_2^0,$$

$$\Phi_2^2 \times \Phi_2^2 = \Phi_0^0 + \Phi_4^0 + \Phi_0^2,$$

$$\Phi_1^1 \times \Phi_2^2 = \Phi_{-1}^1 + \Phi_3^1. \quad (105)$$

The fusion rules imply

$$\Phi_1^1 \Phi_1^1 = \Phi_2^2 + \Phi_2^0,$$

$$(\Phi_1^1 \Phi_1^1)^2 = \Phi_0^0 + 2\Phi_4^0 + 3\Phi_0^2,$$

$$(\Phi_1^1 \Phi_1^1)^3 = 9\Phi_2^2 + 4\Phi_2^0 + 5\Phi_6^0,$$

$$(\Phi_1^1 \Phi_1^1)^4 = 27\Phi_0^2 + 13\Phi_4^0 + 14\Phi_0^0,$$

$$(\Phi_1^1 \Phi_1^1)^5 = 81\Phi_2^2 + 40\Phi_6^0 + 41\Phi_2^0,$$

$$(\Phi_1^1 \Phi_1^1)^6 = 243\Phi_0^2 + 122\Phi_4^0 + 121\Phi_0^0. \quad (106)$$

There appears to be a connection between the Φ_1^1 quasiparticles and the Z_2 vortices. First, notice that one member of a pair of Z_2 vortices should be conjugate to the other member. This is because a pair of Z_2 vortices can be created out of the vacuum on a sphere. Suppose that we identify one member of a pair with the operator $V_\sigma = \Phi_1^1 e^{iQ(1/\sqrt{\nu})\varphi}$ and the other member with its conjugate $V_{\bar\sigma} = \Phi_{-1}^1 e^{-iQ(1/\sqrt{\nu})\varphi}$. From Eq. (106), we see that the number of ways to fuse to the identity for $(V_\sigma V_{\bar\sigma})^n = \Phi_{-2n}^0 (\Phi_1^1 \Phi_1^1)^n$ is as displayed in Table II. Notice

that this agrees exactly with the number of Z_2 invariant states for n Z_2 vortices on a sphere (see Table I).

Notice that the number of ways for $(V_\sigma V_{\bar\sigma})^n$ to fuse to the quasiparticle Φ_4^0 is exactly equal to the number of Z_2 noninvariant states that we obtain from n pairs of Z_2 vortices (see Table II and I). This shows that the Z_2 noninvariant states have a meaning as well. These states carry nontrivial Z_2 charge, so we interpret this as a situation in which there are n pairs of Z_2 vortices and an extra Z_2 charged quasiparticle. The above fusion indicates that we should associate this Z_2 charged quasiparticle to the operator Φ_4^0.

Based on this quantitative agreement between the properties of the Z_2 vortices and results from the Z_4 parafermion FQH state, we conclude that for a pair of Z_2 vortices, one of them should be associated with an operator of the form $\Phi_1^1 e^{iQ(1/\sqrt{\nu})\varphi}$ and the one to which it is connected by a branch cut should be associated with $\Phi_{-1}^1 e^{-iQ(1/\sqrt{\nu})\varphi}$. Furthermore, the possibility of Z_2 noninvariant states should be interpreted as the possibility for the Z_2 vortices to fuse to an electromagnetically neutral Z_2 charged quasiparticle, which we associate with the operator Φ_4^0.

We have not seen how to understand the quantization of electromagnetic charge, Q, for the Z_2 vortices. The external electromagnetic field couples to the field $a^+ = a + \tilde{a}$, so we expect electromagnetically charged quasiparticles to carry flux of the a^+ field. The quantization of charge for the quasiparticles generally arises from the constraint that quasiparticles are mutually local with respect to electrons. We should be able to see how the Z_2 vortices must carry certain quantized units of a^+ flux, but we have not performed this analysis.

VI. CONCLUSION

In this paper, we have computed several topological properties of $U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory and discussed its relation to the Pfaffian and Z_4 parafermion FQH states. For the $l=3$ $U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory, many topological properties agree with those of the Z_4 parafermion state, which strongly suggests that the Chern-Simons theory correctly describes all of the topological properties of this state. This identification also suggests that the phase transition between the $(k,k,k-3)$ bilayer state and the Z_4 parafermion FQH state can be continuous and may, for instance, be described by a Z_2 transition in $2+1$ dimensions.

In the simplest case, for $k=3$, this would be a continuous Z_2 transition at $\nu=2/3$ between the $(3,3,0)$ state and the non-Abelian Z_4 parafermion state. We leave a study of the phase transition itself for future work.

More generally, the methods in this paper may be extended to compute topological properties of Chern-Simons theories with disconnected gauge groups of the form $G \rtimes H$, where G is a connected Lie group and H is a discrete automorphism group of G. There may be other situations also in which an n-layer FQH state passes through a phase transition to an m-layer FQH state, where the Chern-Simons gauge theories for each of the phases will be $G \rtimes H_n$ and $G \rtimes H_m$, respectively, and the phase transition will be described by a discrete gauge symmetry breaking of H_n to H_m. We expect that such a scenario may be possible if the central charges of the corresponding edge theories are the same for the two phases. In this paper, for example, we found that even though there is a phase transition between the $(k,k,k-2)$ bilayer states and the Pfaffian states as the interlayer tunneling is increased, the $l=2$ $U(1) \times U(1) \rtimes Z_2$ theory does not describe the Pfaffian state. In contrast, there is a possible phase transition between the $(k,k,k-3)$ bilayer states and the Z_4 parafermion states, and in this case the $l=3$ $U(1) \times U(1) \rtimes Z_2$ theory does correctly describe the Z_4 parafermion state. One way to understand why simply gauging a Z_2 symmetry does not describe the Pfaffian state is that the central charges of the edge theory changes as the interlayer tunneling is tuned through a phase transition from the bilayer $(k,k,k-2)$ phase to the Pfaffian state, which indicates there is additional physics taking place that this approach does not capture here. The parent bilayer Abelian phase has $c=2$, as does the edge theory of the Z_4 parafermion state, while the edge theory of the Pfaffian state has $c=3/2$.

APPENDIX A: Z_4 PARAFERMION FQH STATES AND PROJECTIVE CONSTRUCTION

Here we explain, from the point of view of a procedure called projective construction,[12] how to understand that the $(k,k,k-3)$ bilayer wave function, upon symmetrization, yields the Z_4 parafermion wave function at $\nu=\frac{2}{2k-3}$, and a different explanation for why we expect that the corresponding Chern-Simons theory should have the gauge group $U(1) \times U(1) \rtimes Z_2$.

In the projective construction approach, one writes the electron operator (which is either bosonic or fermionic, depending on whether we are interested in FQH states of bosons or fermions) in terms of several other fermionic fields, $\psi_1, \cdots \psi_n$, referred to as "partons,"

$$\Psi_e = \psi_{\alpha_1} \cdots \psi_{\alpha_n} C_{\alpha_1 \cdots \alpha_n}, \tag{A1}$$

where $C_{\alpha_1 \cdots \alpha_n}$ are constant coefficients. The continuum field theory that describes interacting electrons in an external

magnetic field can then be rewritten in terms of the partons and a gauge field. The introduction of the partons expands the Hilbert space, so the gauge field is included in order to project the states onto the physical Hilbert space, which is generated by the electron operator. If the partons form a state $|\Phi_{parton}\rangle$, the electron wave function is the projection onto the physical electronic Hilbert space,

$$\Phi_e(z_1, \ldots, z_N) = \langle 0| \prod_i \Psi_e(z_i) |\Phi_{parton}\rangle. \tag{A2}$$

If G is the group of transformations on the partons that keeps the electron operator invariant, then the continuum field theory description will be partons interacting with a gauge field with gauge group G, which ensures that physical excitations, which are created by electron operators, will be singlets of the group G. Since the partons are assumed to form a gapped integer quantum Hall state, they can be integrated out to obtain a Chern-Simons theory with gauge group G.

For example, if we choose the electron operator to be

$$\Psi_{e;1/3} = \psi_1 \psi_2 \psi_3, \tag{A3}$$

then $\Psi_{e;1/3}$ is an $SU(3)$ singlet. If we assume that the partons each form a $\nu=1$ integer quantum Hall state, then the electron wave function is

$$\Phi_{e;1/3}(z_1, \ldots, z_N) = \prod_{i<j} (z_i - z_j)^3, \tag{A4}$$

which is the Laughlin $\nu=1/3$ wave function. The continuum field theory is a theory of three fermions coupled to an $SU(3)$ gauge field. Integrating out the partons will yield a $SU(3)_1$ Chern-Simons theory. This theory is equivalent to the $U(1)_3$ Chern-Simons theory, which is the topological field theory for the $\nu=1/3$ Laughlin state.

If we choose the electron operator to be

$$\Psi_{e;pf} = \psi_1 \psi_2 + \psi_3 \psi_4 \tag{A5}$$

and assume the partons form a $\nu=1$ IQH state, we can obtain the wave function after projection by using the following observation. The $\nu=1$ wave functions are equal to free chiral fermion correlators of a $1+1$-dimensional CFT,

$$\Phi_{\nu=1} = \langle 0| \prod_i \psi(z_i) |\nu=1\rangle = \prod_{i<j} (z_i - z_j)$$
$$\sim \left\langle e^{-iN\phi(z_\infty)} \prod_{i=1}^N \psi(z_i) \right\rangle, \tag{A6}$$

where in the first line, $\psi(z_i)$ is a free fermion operator that annihilates a fermion at position z_i and $|\nu=1\rangle$ is the $\nu=1$ integer quantum Hall state for the fermion ψ; in the second line, $\psi(z_i)$ is interpreted as a free chiral fermion operator in a $1+1$-dimensional CFT and $\frac{1}{2\pi}\partial\phi = \psi^\dagger\psi$ is the density of the fermions. From this, it follows that wave function (A2) with the electron operator $\Psi_{e;pf}$ can be obtained by taking the correlator

$$\Phi_{e;pf} \sim \left\langle e^{-iN\phi(z_\infty)} \prod_{i=1}^{N} \Psi_{e;pf}(z_i) \right\rangle, \quad (A7)$$

where $\Psi_{e;pf} = \psi_1\psi_2 + \psi_3\psi_4$ and $\psi_i(z)$ now interpreted as a free complex chiral fermion in a $1+1$-dimensional CFT. The operator product algebra generated by the electron operator in this case can be checked to be reproduced if we instead write the electron operator as

$$\Psi_{e;pf} = \psi\eta, \quad (A8)$$

where η is a Majorana fermion and ψ is a free complex chiral fermion. The correlation function with N insertions of this operator is known to yield the $\nu = 1$ Pfaffian wave function. The gauge group that keeps $\Psi_{e;pf}$ invariant is $SO(5)$, and thus the Chern-Simons theory for the $\nu = 1$ Pfaffian is a $SO(5)_1$ Chern-Simons theory.[12]

Now consider a bilayer wave function, where we have two electron operators, one for each layer, and the wave function is given by

$$\Phi(\{z_i\}, \{w_i\}) \sim \left\langle e^{-iN\phi(z_\infty)} \prod_{i=1}^{N} \Psi_{e1}(z_i)\Psi_{e2}(w_i) \right\rangle. \quad (A9)$$

The single-layer wave function that can be obtained by symmetrizing or antisymmetrizing over the electron coordinates in the two layers can be obtained by choosing the single-layer electron operator to be $\Psi_e = \Psi_{e1} + \Psi_{e2}$,

$$\Phi(\{z_i\}) = S\{\Phi(\{z_i\}, \{w_i\})\}$$
$$\sim \left\langle e^{-iN\phi(z_\infty)} \prod_{i=1}^{2N} [\Psi_{e1}(z_i) + \Psi_{e2}(z_i)] \right\rangle, \quad (A10)$$

where we have set $z_{N+i} = w_i$.

In the case of the Pfaffian, this shows us that the $(2, 2, 0)$ state, when symmetrized, yields the Pfaffian wave function. If we instead consider $\Psi_{e1} = \psi_1\psi_2\psi_3$ and $\Psi_{e2} = \psi_4\psi_5\psi_6$, we obtain the $(3, 3, 0)$ state. The $(3, 3, 0)$ state, when symmetrized, will therefore be given by

$$\Phi(\{z_i\}) \sim \left\langle e^{-iN\phi(z_\infty)} \prod_i \Psi_e(\{z_i\}) \right\rangle, \quad (A11)$$

with $\Psi_e = \psi_1\psi_2\psi_3 + \psi_4\psi_5\psi_6$. It can be checked that the operator product algebra generated by this electron operator is also generated by the operator $\Psi_e = \Phi_2^0 e^{i\sqrt{3/2}\phi}$, where Φ_2^0 is a simple-current operator in the Z_4 parafermion CFT and ϕ is a scalar boson. Thus, this wave function is the wave function of the Z_4 parafermion FQH state at $\nu = 2/3$. Furthermore, the gauge group that keeps the electron operator invariant is $SU(3) \times SU(3) \rtimes Z_2$, so we expect that the corresponding Chern-Simons theory for this phase should be $SU(3)_1 \times SU(3)_1 \rtimes Z_2$ Chern-Simons theory, which we expect to be equivalent to $U(1)_3 \times U(1)_3 \rtimes Z_2$ Chern-Simons theory. One would then guess that the generalization to the $(k, k, k-3)$ states and the $\nu = \frac{2}{2k-3} Z_4$ parafermion states is the $U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory described in this paper.

APPENDIX B: MORE DETAILED DISCUSSION OF THE GROUND-STATE DEGENERACY

Here we like to discuss the ground-state degeneracy of the $U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory in more detail. For $l = 2$, the filling fraction is $\nu = \frac{1}{k-1}$ and formula (32) gives $9(k-1)$ states on a torus. Compare this to the torus degeneracy of the $\nu = \frac{1}{k-1}$ Pfaffian state, which is $3(k-1)$. We see that the $U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory for $l = 2$ has a torus ground-state degeneracy that is three times that of the Pfaffian state. The origin of this factor of 3 can be thought of in the following way. It is known that $O(2)_{2l}$ Chern-Simons theory has $l+7$ ground states[21] (see Appendix C). So, $U(1)_{k-1} \times O(2)_4$ has $9(k-1)$ ground states on a torus. Furthermore, the gauge group $U(1) \times O(2)$ is similar to $U(1) \times U(1) \rtimes Z_2$ if one considers the positive and negative combinations of the two $U(1)$ gauge fields: if one considers $a^+ = a + \tilde{a}$ and $a^- = a - \tilde{a}$, the gauge group can be thought of as $U(1) \times O(2)$ because the action of the Z_2 is to take $a^- \rightarrow -a^-$. Now, $O(2)$ Chern-Simons theory at level $2l$ is known to correspond to the Z_2 rational orbifold conformal field theory at level $2l$, which for $l = 2$, is known to be dual to two copies of the Ising CFT.[20,21] The Ising CFT has three primary fields, and the CFT corresponding to the Pfaffian is one that contains an Ising CFT and a $U(1)$ CFT. In this sense our theory has an extra copy of the Ising model, which accounts for the extra factor of 3 in the torus degeneracy. We can see this another way by noticing that the central charge of the Ising CFT is $1/2$ and the central charge of the CFT that corresponds to the Pfaffian state is $c = 3/2$. Meanwhile, the CFT corresponding to the $U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory has $c = 2$, which corroborates the fact that it has an extra copy of the Ising model.

For $l = 3$, the filling fraction is $\nu = \frac{2}{2k-3}$ and Eq. (32) gives $5(2k-3)$ ground states on a torus. Compare this to the $\nu = \frac{2}{2k-3} Z_4$ parafermion state, which also has a torus degeneracy of $5(2k-3)$. This might be expected from the fact that $O(2)_{2l}$ Chern-Simons theory corresponds to the Z_4 parafermion CFT when $l = 3$. However, there is a crucial issue that needs to be addressed here. In the case $l = 2$, we could see that $U(1)_{k-1} \times O(2)_4$ Chern-Simons theory gives the same number of ground states on a torus as the $U(1) \times U(1) \rtimes Z_2$ theory did, implying that we could perhaps think of the $U(1)$ sector of the theory as separate from the $O(2)$ sector. This fails in the $l = 3$ case. We would be tempted to write $U(1)_{k-3/2} \times O(2)_6$ because $(k-3/2) \times (3+7)$ gives the right ground-state degeneracy. This fails because the ground state degeneracy of $U(1)_{k-3/2}$ Chern-Simons theory is not $(k-3/2)$. $U(1)_q$ Chern-Simons theory is typically defined to have integer q, but the quantization procedure may also be applied in cases where q is not an integer. In these latter cases, the quantum states do not transform as a one-dimensional representation under large gauge transformations. One may wish to reject a theory in which the quantum states are not gauge invariant, in which case $U(1)_q$ is not defined for noninteger q. On the other hand, if these situations are allowed, then it can be shown that $U(1)_q$ Chern-Simons theory, for $q = p/p'$ (where p and p' are coprime), has a torus degeneracy of pp'.[30] Therefore, $U(1)_{k-3/2}$ Chern-

Simons theory, to the extent that it is well defined, has degeneracy $2(2k-3)$. In either case, it is clear that the $U(1)$ and $O(2)$ sectors cannot be disentangled and that the correct definition of the theory is the $U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory presented here. To summarize, for $l=2$, $U(1) \times U(1) \rtimes Z_2$ Chern-Simons theory describes the Pfaffian state but with an extra copy of the Ising model, while for $l=3$, the $U(1) \times U(1) \rtimes Z_2$ theory gives the same ground-state degeneracy as the Z_4 parafermion quantum Hall state.

APPENDIX C: $O(2)$ CHERN-SIMONS THEORY AND Z_2 RATIONAL ORBIFOLD CONFORMAL FIELD THEORIES

Here we summarize previously known results from $O(2)$ Chern-Simons theory and the Z_2 orbifold CFT and apply the Z_2 vortex analysis of this paper to the $O(2)$ Chern-Simons theory. Moore and Seiberg[21] first discussed Chern-Simons theories with disconnected gauge groups of the form $G \rtimes P$, where G is a connected group with a discrete automorphism group P, and the connection of these Chern-Simons theories to G/P orbifold conformal field theories. As a special example, they discussed the case where $G=U(1)$ and $P=Z_2$. In the two-dimensional conformal field theory, this is known as the Z_2 orbifold and it was explicitly analyzed in Ref. 20. It is the theory of a scalar boson φ compactified at a radius R so that $\varphi \sim \varphi + 2\pi R$, and with an additional Z_2 gauge symmetry: $\varphi \sim -\varphi$.

1. Z_2 rational orbifold CFT

When $\frac{1}{2}R^2$ is rational, i.e., $\frac{1}{2}R^2 = p/p'$, with p and p' coprime, then it is useful to consider an algebra generated by the fields $j = i\partial\varphi$, and $e^{\pm i\sqrt{2N}\varphi}$ for $N = pp'$. This algebra is referred to as an extended chiral algebra. The infinite number of Virasoro primary fields in the $U(1)$ CFT can now be organized into a finite number of representations of this extended algebra $\mathcal{A}$. There are $2N$ of these representations, and the primary fields are written as $V_l = e^{il\varphi/\sqrt{2N}}$, with $l = 0, 1, \ldots, 2N-1$.

In the Z_2 orbifold, one now considers representations of the smaller algebra $\mathcal{A}/Z_2$. This includes the Z_2 invariant combinations of the original primary fields, which are of the form $\cos(l\varphi/\sqrt{2N})$; there are $N+1$ of these. In addition, there are six new operators. The gauging of the Z_2 allows for twist operators that are not local with respect to the fields in the algebra $\mathcal{A}/Z_2$, but rather local up to an element of Z_2. It turns out that there are two of these twisted sectors, and each sector contains one field that lies in the trivial representation of the Z_2, and one field that lies in the nontrivial representation of Z_2. These twist fields are labeled σ_1, τ_1, σ_2, and τ_2. In addition to these, an in-depth analysis[20] shows that the fixed points of the Z_2 action in the original $U(1)$ theory split into a Z_2 invariant and a noninvariant field. We have already counted the invariant ones in our $N+1$ invariant fields, which leaves two new fields. In total, there are $N+7$ fields in the Z_2 rational orbifold at "level" $2N$.

The dimension of the space of conformal blocks on a genus g surface is given by the following formula:[26]

TABLE III. Primary fields in the Z_2 orbifold for $N=2$, their scaling dimensions, and the fields from Ising^2 to which they correspond.

Z_2 orb. field	Scaling dimension, h	Ising^2 fields
1	0	$\mathbb{I} \otimes \mathbb{I}$
j	1	$\psi \otimes \psi$
ϕ_N^1	$1/2$	$\mathbb{I} \otimes \psi$
ϕ_N^2	$1/2$	$\psi \otimes \mathbb{I}$
ϕ_1	$1/8$	$\sigma \otimes \sigma$
σ_1	$1/16$	$\sigma \times \mathbb{I}$
σ_2	$1/16$	$\mathbb{I} \otimes \sigma$
τ_1	$9/16$	$\sigma \otimes \psi$
τ_2	$9/16$	$\psi \otimes \sigma$

$$\dim V_g = \mathrm{Tr}\left(\sum_{i=0}^{N-1} N_i^2\right)^{g-1} = \sum_{n=0}^{N-1} S_{n0}^{-2(g-1)}. \quad (C1)$$

The S matrix was computed for the Z_2 orbifold in Ref. 20, so we can immediately calculate the above quantity in this case. The result is

$$\dim V_g = 2^{g-1}[2^{2g} + (2^{2g} - 1)N^{g-1} + N^g]. \quad (C2)$$

For $N=2$, it was observed that the Z_2 orbifold is equivalent to two copies of the Ising CFT. For $N=3$, it was observed that the Z_2 orbifold is equivalent to the Z_4 parafermion CFT of Zamolodchikov and Fateev.[29]

In Tables III and IV we list the fields from the Z_2 orbifold for $N=2$ and $N=3$, their scaling dimensions, and the fields in the Ising^2 or Z_4 parafermion CFTs that they correspond to.

2. $O(2)$ Chern-Simons theory on a torus

The claim of Moore and Seiberg was that $O(2)$ Chern-Simons theory at level $2N$ corresponds to the Z_2 rational orbifold CFT at level $2N$. The first step in showing this is to show that the degeneracy of this theory on a torus is $N+7$.

TABLE IV. Primary fields in the Z_2 orbifold for $N=3$, their scaling dimensions, and the Z_4 parafermion fields that they correspond to.

Z_2 orb. field	Scaling dimension, h	Z_4 parafermion field
1	0	Φ_0^0
j	1	Φ_4^0
ϕ_N^1	$3/4$	Φ_2^0
ϕ_N^2	$3/4$	Φ_6^0
ϕ_1	$1/12$	Φ_2^2
ϕ_2	$1/3$	Φ_0^2
σ_1	$1/16$	Φ_1^1
σ_2	$1/16$	Φ_{-1}^1
τ_1	$9/16$	Φ_3^1
τ_2	$9/16$	Φ_5^1

This is done in the following way. The classical configuration space of pure Chern-Simons theory with gauge group G consists of flat G bundles on a torus. Flat $O(2)$ bundles can be split into two classes, those that can be considered to be $SO(2) = U(1)$ bundles, and those that cannot. In the first case, we simply need to take the space of states in $U(1)_{2N}$ Chern-Simons theory and keep the Z_2 invariant states. This leaves $N+1$ states.

In addition to these, there are flat, twisted bundles. Flat bundles are classified by $\hom[\pi_1(M) \to G]/G$. This is the space of homomorphisms of the fundamental group of the manifold M into the gauge group G, modulo G. Let us study the space of flat twisted $O(2)$ bundles. We first write the gauge field as

$$A_\mu = \begin{pmatrix} a_\mu & 0 \\ 0 & -a_\mu \end{pmatrix}. \qquad (C3)$$

The group is composed of $U(1)$ elements, which we write in terms of the Pauli matrix σ_3: $e^{i\alpha\sigma_3}$. We write the Z_2 element as the Pauli matrix σ_1. The Z_2 action is therefore $A_\mu \to \sigma_1 A_\mu \sigma_1 = -A_\mu$. We can write a Lagrangian for this theory,

$$L = \frac{2N}{4\pi} \int_M d^2x \, a\partial a. \qquad (C4)$$

We are concerned with the case where M is the torus, T^2. $\pi_1(T^2)$ is generated by two elements, a and b, the two non-contractible loops of the torus. We must study the homomorphism $h : \pi_1(T^2) \to G$. $\pi_1(T^2)$ is an Abelian group, and since h is a homomorphism, we must have

$$h(\alpha a + \beta b) = \alpha\beta h(a)h(b) = \alpha\beta h(b)h(a). \qquad (C5)$$

Suppose we are twisted in the a direction only. Then, we have

$$h(a) = \sigma_1 e^{i\theta\sigma_3}, \quad h(b) = e^{i\phi\sigma_3}. \qquad (C6)$$

Modding out by the group $O(2)$, we find that $\theta \sim -\theta + 2\pi m \sim \theta + 2\alpha$ for any α. The first equivalence comes from modding out by the Z_2 element, while the second element comes from modding out by the $U(1)$ element. Similarly, $\phi \sim -\phi + 2\pi n$. n and m are integers. The constraint $h(a)h(b) = h(b)h(a)$ further implies that $\phi = 0$ or π. The distinct solutions to these relations are therefore that

$$(\theta, \phi) = (0, \pi) \text{ or } (0,0). \qquad (C7)$$

A similar analysis shows that the cases in which the bundle is twisted in the b direction only or along both a and b also each admit only two distinct bundles. Therefore, there are a total of six distinct twisted flat $O(2)$ bundles. Each corresponds to a single quantum state for a total of $N+7$ states in the $O(2)$ Chern-Simons theory on a torus.

3. Z_2 vortices in $O(2)$ Chern-Simons theory

This section is essentially an application of the analysis of Z_2 vortices in the case of $G = U(1) \times U(1) \rtimes Z_2$ to the case $G = O(2)$. In this case, a Z_2 vortex takes the gauge field to minus itself. With n pairs of Z_2 vortices, we again deform the manifold on which the gauge field A_μ is defined, consider a double copy, and glue the two copies together to obtain a genus $g = n-1$ surface.

The analog of Eq. (93) in this case is

$$c_\mu(x,y) = \begin{cases} a_\mu(x,y), & x \leq 0 \\ a_\mu[R(x,y)], & x \geq 0, \end{cases} \qquad (C8)$$

and in terms of c_μ we immediately see that the action is that of a $U(1)$ Chern-Simons theory at level N,

$$L = \frac{2N}{4\pi} \int_{M_1} d^2x \, a\partial a = \frac{N}{4\pi} \int_M d^2x \, c\partial c. \qquad (C9)$$

On a genus $g = n-1$ surface, there are $|N|^{n-1}$ states. But we need to project onto the Z_2 invariant sector. The action of the Z_2 is to take $c \to -c$. We count $(|N|^{n-1}+1)/2$ Z_2 invariant states when N is odd. If $N=2$, all of the N^{n-1} states are Z_2 invariant.

How does this relate to the corresponding conformal field theory, which is the Z_2 rational orbifold at level $2N$? Let us examine a few cases. When $N=1$, this theory is the same as $U(1)_8$ CFT, which is abelian and which therefore should have degeneracy 1 for all n.

When $N=2$, the orbifold CFT is the same as two copies of the Ising CFT. The Ising CFT has a single non-Abelian field, σ. The space of conformal blocks corresponding to $2n$ σ fields on a sphere in the Ising CFT is 2^{n-1}, which agrees with our above analysis for $N=2$. However, a theory with two copies of the Ising CFT would have many non-abelian fields:

$$\sigma \otimes \mathbb{I}, \quad \sigma \otimes \psi,$$

$$\mathbb{I} \otimes \sigma, \quad \psi \otimes \sigma,$$

$$\sigma \otimes \sigma. \qquad (C10)$$

The space of conformal blocks corresponding to $2n$ of either $\sigma \otimes \mathbb{I}$, $\sigma \otimes \psi$, $\mathbb{I} \otimes \sigma$, or $\psi \otimes \sigma$ will have dimension 2^{n-1}. However, the dimension of the space of conformal blocks corresponding to $2n$ $\sigma \otimes \sigma$ fields will be different. Thus Z_2 vortices in the $O(2)$ Chern-Simons with $N=2$ are closely related to the fields $\sigma \otimes \mathbb{I}$, $\sigma \otimes \psi$, $\mathbb{I} \otimes \sigma$, and $\psi \otimes \sigma$.

When $N=3$, the orbifold CFT is dual to the Z_4 parafermion CFT of Zamolodchikov and Fateev. We expect the Z_2 vortices to correspond to the Φ_1^1 fields, and in fact we obtain the correct number of states in the presence of n pairs of Z_2 vortices, as discussed earlier.

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